Turbulent Kinetic Energy Budget in the Near-Wall Region

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From a knowledge of the measured budget of the temperature variance in the near-wall region of a turbulent boundary layer, the behavior of the turbulent kinetic energy diffusion is obtained by analogy to that for the temperature variance diffusion. Using available data for the turbulent energy dissipation and the molecular diffusion of kinetic energy, the pressure diffusion is then inferred by difference. The magnitude of the pressure diffusion is small compared with other terms in the energy budget. The sum of the pressure diffusion and turbulent kinetic energy diffusion terms is inadequately described by the model used in the k-ε calculation.

Introduction

THE modeling of turbulent shear flows in the immediate neighborhood of a solid boundary has made extensive use of the wall function approach,[1] but there have also been several calculations where the numerical computations are extended through to the wall. Second-order models that have directly taken into account the near-wall region have included k-ϵ models,[2-6] with modeled transport equations for the turbulent kinetic energy k and the average energy dissipation ϵ, and Reynolds-stress models,[7-8] with modeled transport equations for the Reynolds stresses.

Patel et al.[9] assessed the performance of several k-ϵ models for near-wall and low Reynolds number flows. They found that relatively few models yielded satisfactory results but noted that even those models needed further refinement before they could be used with confidence. Any refinement would, however, require that accurate data be available for the near-wall region. For example, the modeled transport equation for k in the k-ϵ mode would require reliable near-wall distributions for turbulent diffusion and for energy dissipation. In a recent paper, Bernard[10] compared the measured turbulent kinetic energy in a channel flow with predictions of several near-wall forms of the k-ϵ model. All forms underestimate the measured peak k value, after allowance is made for the experimental scatter and the possible Reynolds number dependence of this quantity. Bernard attributed the discrepancy to the incorrect behavior of the modeled pressure diffusion term in the k equation. There are, however, no measurements of the pressure diffusion term while available measurements of the k diffusion term and the dissipation term are at best only incomplete. By contrast, it is possible to obtain the majority of the terms in the transport equation for $\overline{\theta^2}/2$, where $\overline{\theta^2}$ is the temperature variance in the near-wall region. Such measurements were recently made by Krishnamoorthy and Antonia[11] in a turbulent boundary layer. In particular, all components of the temperature dissipation ϵ_θ were measured and, in the absence of a pressure term, a relatively accurate estimate was obtained, by difference, for the turbulent diffusion of $\overline{\theta^2}/2$. In the present paper, an estimate for the turbulent diffusion of k in the near-wall region is inferred by analogy to the behavior of the turbulent diffusion $\overline{\theta^2}/2$. Using existing measurements of ϵ and of the molecular diffusion of k, a plausible distribution is obtained for the pressure diffusion term. The distribution for the total diffusion term is compared with the form used in k-ϵ calculations.

Near-Wall Budget of $\overline{\theta^2}/2$

A close approximation to the transport equation of $\overline{\theta^2}/2$ in the near-wall region is given by

$$Pr^{-1}(\overline{\theta^{+2}}/2)_{,22} - [\overline{u_2^+(\theta^{+2}/2)}]_{,2} - \overline{u_2^+\theta^+}T^+_{,2} - \epsilon_\theta^+ = 0 \qquad (1)$$

where u_2 is the velocity fluctuation normal to the wall, T the local mean temperature relative to ambient, Pr the molecular Prandtl number ($\equiv \nu/\alpha$, where ν and α are the molecular diffusivities of momentum and heat, respectively), $(\ )_{,2} \equiv \partial(\ )/\partial x_2^+$, and the superscript + denotes normalization by wall variables (the friction velocity U_τ, the friction termperature θ_τ, and the length scale ν/U_τ). The terms on the left-hand side of Eq. (1) represent molecular diffusion, turbulent diffusion, production and dissipation of $\overline{\theta^2}/2$, respectively. The turbulent diffusion was inferred by difference. The molecular diffusion and dissipation were estimated from measurements of $\overline{\theta^2}$ and ϵ_θ. The production was estimated from measurements of T and a distribution of $\overline{u_2\theta}$, calculated with the mean enthalpy equation using measurements of T and of the mean velocity U_1. Experimental details used to construct the budget are given in Ref. 11

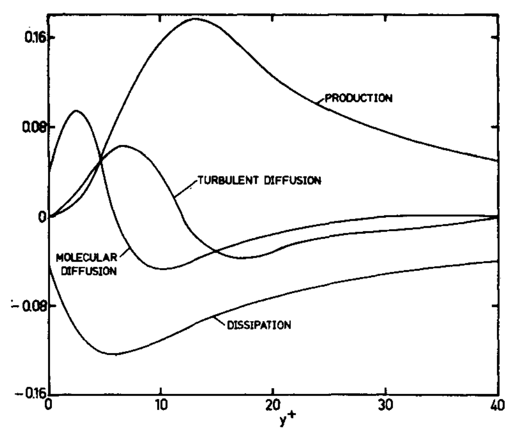

Fig. 1 Budget of $\overline{\theta^2}/2$ in near-wall region (measurements of Ref. 11).

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and are not repeated here. Distributions of the four terms on the left-hand side of Eq. (1) are shown in Fig. 1 for the region $0 \le x_2^+ \le 40$.

The behavior of the molecular diffusion and turbulent dissipation terms at or near $x_2^+ = 0$ is worth noting. These terms satisfy an important boundary condition at $x_2^+ = 0$

$$Pr^{-1}(\overline{\theta^{+2}}/2)_{,22} = \epsilon_\theta^+ \tag{2}$$

that is the limiting form of Eq. (1) at $x_2^+ = 0$. The identity $(\overline{\theta^{+2}}/2)_{,22} \equiv (\theta_{,2}^+)^2$, valid at $x_2^+ = 0$, confirms Eq. (2). In the limit of $x_2^+ \to 0$, it is easy to show, using Taylor series expansions, that the turbulent diffusion and production terms vary as x_2^{+3}, whereas magnitudes of molecular diffusion and dissipation terms increase in proportion of x_2^+ from their respective values at the wall. This relative behavior is reflected in the distributions of Fig. 1 in the limit of $x_2^+ \to 0$.

The distribution (Fig. 1) of the turbulent diffusion (obtained by difference) indicates a contribution of the same sign as the production for $x_2^+ \le 12$. This positive contribution is offset by a negative contribution in the range $12 \le x_2^+ \le 40$, so that

$$\int_0^{x_2^+ \simeq 40} \overline{(u_2^+ \theta^{+2})}_{,2} \mathrm{d}x_2^+ \simeq 0 \tag{3}$$

Between $x_2^+ \simeq 40$ and $x_2 \simeq \delta$ the turbulent diffusion is not zero, its variation, already reported in Ref. 11, being similar to that obtained (e.g., Ref. 12) for the turbulent kinetic energy diffusion. In particular, the turbulent diffusion and the advection are approximately in balance and become more important than the production or dissipation near the edge of the layer. In the region that extends from $x_2^+ \simeq 40$ and $x_2 \simeq \delta$, the turbulent diffusion term changes sign and its variation is such that

$$\int_{x_2^+ \simeq 40}^{x_2 \simeq \delta} \overline{(u_2 \theta^2)}_{,2} \mathrm{d}x_2 \simeq 0 \tag{4}$$

The addition of Eqs. (3) and (4) satisfies the requirement

$$\int_0^\delta \overline{(u_2 \theta^2)}_{,2} \mathrm{d}x_2 \simeq 0 \tag{5}$$

It should be noted that the magnitude of the turbulent diffusion is significantly larger in the near-wall region than in the outer layer. In this context, validation of Eq. (3), which is essentially equivalent to validating Eq. (5), is a useful indicator of the accuracy of the measured terms of Eq. (1) in the near-wall region.

Near-Wall Budget of k

A close approximation to the transport equation of k in the near-wall region is given by

$$k_{,22}^+ - \overline{(u_2{}^+ p^+)}_{,2} - \overline{(u_2^+ k^+)}_{,2} - \overline{u_1^+ u_2^+}\, U_{1,2}^+ - \epsilon^+ = 0 \tag{6}$$

where p is the kinematic pressure. The terms on the left-hand side of Eq. (6) represent molecular diffusion, turbulent diffusion of p, turbulent diffusion of k, production and dissipation of k, respectively. Except for the pressure term, the other terms in Eq. (6) are analogous to those in Eq. (1).

The general difficulties of obtaining an accurate budget of k near the wall are well known and are, therefore, only touched on here. Little is known about the pressure term, apart from its limiting behavior at the wall,[13] and there is also some doubt about the behavior of the third term in Eq. (6). It has not been possible to measure all the components of ϵ. Laufer[14] estimated five of them and assumed isotropy to estimate the remaining four. The latter assumption is tenuous, as demonstrated in Ref. 11 in context of the components of ϵ_θ.

Bernard and Berger[13] improved the reliability of the budget of k in the near-wall region by combining the use of Taylor series expansions in the region $0 \le x_2^+ \le 2$ with available data[12] for $x_2^+ \ge 7$. In the region $2 \le x_2^+ \le 7$, curves were faired-in for the molecular diffusion, dissipation, and pressure diffusion. For the present purpose, their distributions of molecular diffusion and dissipation have been retained. They are reproduced in Fig. 2, the dashed lines corresponding to the "faired-in" transition regions of Ref. 13. The near-wall budget in Refs. 10 and 13 are shown only up to $x_2^+ \simeq 10$; for $x_2^+ > 10$, we have used Townsend's[12] distributions for molecular diffusion and dissipation. Distribution in Fig. 2 of the turbulent diffusion of k is based on the *assumption* that the turbulent diffusions of the scalar quantities $\overline{\theta^2}/2$ and k should be qualitatively similar in the near-wall region. In support of this assumption are: 1) flow visualization evidence[15] that suggests a close similarity between velocity and thermal structures; 2) analogy between corresponding terms in Eqs. (1) and (6); 3) the close similarity between the distributions for molecular diffusion, production, and dissipation in Figs. 1 and 2; and 4) the near equality (Ref. 1), to within the uncertainty in ϵ, between the time scale k/ϵ of velocity field and time scale $\overline{\theta^2}/2\epsilon_\theta$ of the temperature field.

The particular distribution for $\overline{(u_2^+ k^+)}_{,2}$, shown in Fig. 2, was estimated from

$$\frac{\overline{(u_2^+ k^+)}_{,2}}{[\overline{u_2^+ (\theta^{+2}/2)}]_{,2}} = \frac{\overline{u_1{}^+ u_2^+}\, U_{1,2}^+}{\overline{u_2^+ \theta^+}\, T_{,2}^+} \tag{7}$$

Clearly, the accuracy with which $\overline{(u_2^+ k^+)}_{,2}$ can be obtained depends mainly on the accuracy of the denominator on the

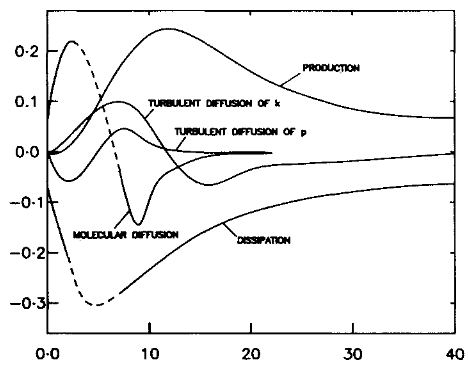

Fig. 2 Budget of k in near-wall region.

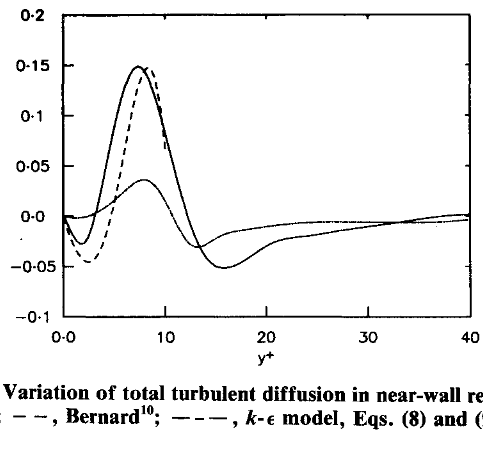

Fig. 3 Variation of total turbulent diffusion in near-wall region: —, present; – –, Bernard[10]; — — —, k-ϵ model, Eqs. (8) and (9).

left side of Eq. (7) and, therefore, on the accuracy (see Ref. 11) with which ϵ_θ is estimated. The production terms were chosen for the above scaling because of the accuracy with which they can be estimated. However, an almost identical distribution of $\overline{(u_2^+ k^+)}_{,2}$ was obtained by using a scaling based on dissipation, i.e., by replacing the right side of Eq. (7) by $\epsilon^+/\epsilon_\theta^+$.

The pressure diffusion term, shown in Fig. 2, is estimated by difference. Near $x_2^+ = 0$, this term is negative, its magnitude increasing linearly away from the wall, as required by its limiting wall behavior.[13] The magnitude of the pressure diffusion is small compared with other terms in Eq. (6) in the region $0 < x_2^+ \leq 12$ and significantly smaller than Townsend's[12] estimate of this term. The present distribution of $\overline{(u_2^+ p^+)}_{,2}$ satisfies the integral

$$\int_0^{x_2^+ \simeq 40} \overline{(u_2^+ p^+)}_{,2} \mathrm{d}x_2^+ = 0$$

approximately. The difference between the present diffusion distribution and that obtained by Townsend[12] or Bernard and Berger[13] is mainly due to the difference between the k diffusion used by these authors and the present distribution of $-\overline{(u_2^+ k^+)}_{,2}$. In Townsend's[12] budget, $-\overline{(u_2^+ k^+)}_{,2}$ is negative throughout the near-wall region; with the obvious qualification that measurements of this term are of limited accuracy, Laufer's[14] data suggested that this term is positive for $x_2^+ \leq 6$. Moin and Kim's[16] numerical calculation indicated that this term is positive for $x_2^+ \leq 15$ and negative at larger values of x_2^+, in qualitative agreement with the distribution in Fig. 2. The present positive value, near $x_2^+ = 0$, of $-\overline{(u_2^+ k^+)}_{,2}$ suggests that $-\overline{u_2^+ k^+}$ initially increases away from the wall. A positive correlation between $-u_2$ and k (or θ^2) seems plausible if it is associated with the dominance in this region of fluid moving towards the wall ($u_2 < 0$) with an excess of turbulent energy or temperature variance. There is some evidence[17,18] that the skewness of u_2 becomes negative very near the wall. Near-wall measurements made in our laboratory indicate that the correlation $\overline{u_1\theta^2}$ increases from zero to the wall to a maximum at $x_2^+ \simeq 15$; this trend supports the inferred distributions of $-[\overline{u_2^+(\theta^{+2}/2)}]_{,2}$ in Fig. 1 and of $-\overline{(u_2^+ k^+)}_{,2}$ in Fig. 2.

It is of interest to examine the behavior of the total diffusion, or sum, of the pressure diffusion and kinetic energy diffusion, since it is this sum that is modeled in the transport equation for k in k-ϵ models, viz.

$$-\overline{(u_2^+ p^+)}_{,2} - \overline{(u_2^+ k^+)}_{,2} = (\nu_T^+ k_{,2}^+)_{,2} \tag{8}$$

with the eddy viscosity ν_T generally assumed given by

$$\nu_T = C_\mu f_\mu \frac{k^2}{\epsilon_d} \tag{9}$$

where C_μ is a constant (= 0.09), f_μ a damping function to account for the influence of the wall, and $\epsilon_d = \epsilon - 2\nu\,(\sqrt{k}_{,2})^2$. The present total diffusion, shown in Fig. 3, is qualitatively similar to Bernard's[10] distribution, despite important differences between the present distributions and those of Bernard for $-\overline{(u_2^+ p^+)}_{,2}$ and especially $-\overline{(u_2^+ k^+)}_{,2}$. The behavior of the total diffusion near $x_2^+ = 0$ essentially mirrors behavior of the pressure diffusion term. The modeled total diffusion shown in Fig. 3 is for $f_\mu = 1 - \exp(-0.0115 x_2^+)$, the particular form used by Bernard.[10] Although the gradient-type total diffusion, given by Eqs. (5) and (6), satisfies reasonably well the constraint

$$\int_0^{x_2^+ \simeq 40} (\nu_T^+ k_{,2}^+)_{,2} \mathrm{d}x_2^+ = 0$$

it is an inadequate model for the present total diffusion, especially close to the wall. This inadequacy may be partly responsible for failure of the k-ϵ calculation in estimating maximum value of k. Since the near-wall dissipation is also incorrectly calculated[10] by the k-ϵ model, the possibility that the ϵ transport equation is incorrectly modeled in the near-wall region cannot be ruled out.

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